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MULTIPLE SOLUTIONS OF A FREE BOUNDARY FRC EQUILIBRIUM PROBLEM IN A METAL CYLINDER

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I. INTRODUCTION

Field reversed theta pinch experiments routinely produce very prolate plasma equilibria; magnetohydrodynamic equilibrium codes do not. Since the experimental plasmas seem to be stable against the tilting-instability that is predicted for moderately elongated equilibria, it has been conjectured that the observed stability is due to exaggerated elongation. It has been difficult to test this hypothesis because long equilibria have been difficult to compute. Previous calculations have indicated that such equilibria might exist[1-4], but none of these computed equilibria has been completely satisfactory.

We present a new approach to the computation of FRC equilibria that avoids the previously encountered difficulties. For arbitrary pressure profiles it is computationally expensive, but for one special pressure profile the problem is simple enough to require only minutes of Cray time; it is this problem that we have solved. We solve the Grad-Shafranov equation, $\Delta^*\psi = -r^2p'(\psi)$, in an infinitely long flux conserving cylinder of radius a with the boundary conditions that $\psi(a,z) = -\psi_w$ and that $\partial\psi/\partial z = 0$ as $|z|$ approaches infinity. The pressure profile is $p'(\psi) = cH(\psi)$ where c is a constant and where $H(x)$ is the Heaviside function. We have found four solutions to this problem: There is a purely vacuum state, two z-independent plasma solutions, and an r-z-dependent plasma state. These last three solutions are obtained only if the constant c is greater than a certain value; for c smaller than this lower limit the three plasma solutions cease to exist. At the critical value of c all three plasma solutions coalesce, and it is near this critical value of c that elongated equilibria are obtained. This means that the elongated equilibria are found near a bifurcation point of the solution set, a notoriously difficult region in which to compute. This probably explains why elongated solutions have been so difficult to find.

II. ONE-DIMENSIONAL SOLUTIONS

Except for the trivial vacuum solution, $\psi = -\psi_w r^2/a^2$, the z-independent solutions are the simplest to find. An elementary calculation yields

$$\psi = \begin{cases} \frac{1}{2}r^2\left(b - \frac{1}{4}cr^2 \right), & r < r_{sep} \\ -\frac{1}{2}br^2 + d, & r_{sep} < r < a \end{cases} \tag{1}$$

where $$b = \frac{a^2c}{8}\left[1 \pm \left(1 - \frac{32\psi_w}{ca^4} \right)^{1/2} \right], \tag{2}$$

and where $d = 2b^2/c$ and $\psi = 0$ at $r_{sep} = 2\sqrt{b/c}$. Note that there are two possible solutions; these two solutions are realized only if b is real, i.e. only if $c \geq 32\psi_w/a^4$. Figure 1 displays the $\psi = 0$ radius as a function of c. The upper solution in this figure is a high trapped flux solution that compresses the vacuum flux against the wall as c becomes large. The lower solution in this figure is a low trapped flux solution that is squeezed to the axis by the vacuum flux as c becomes large. The presence of a value of c below which no equilibria exist is explained by noting that the toroidal current density is given by $j_\theta = -crH(\psi)$. If c is too small, there is not enough current density to produce field reversal, and no solutions of the model problem are possible. This same argument should apply to any two-dimensional solutions as well, so we expect in any family of equilibria parameterized by c to encounter a lower limit in c below which no equilibria exist.

III. A TWO-DIMENSIONAL SOLUTION

From studies of Hill's vortex equilibria done by us (and independently by John Boyd), we know there is at least one two-dimensional family of solutions parametrized by c. With ψ given by the Hill's vortex formula inside the separatrix, a matching vacuum field outside the separatrix may be constructed by means of ellipsoidal coordinates. For both prolate and oblate Hill's vortices, the matching vacuum field has mirror coils at infinity, but for the spherical Hill's vortex, the field lines at infinity are straight. This spherical solution is given by

$$\psi = \begin{cases} \frac{3}{4} B_0 r^2 \left(1 - \frac{r^2+z^2}{\rho_0^2} \right), & r^2+z^2 < \rho_0^2 \\ -\frac{1}{2} B_0 r^2 \left[1 - \left(\frac{\rho_0^2}{r^2+z^2} \right)^{3/2} \right], & r^2+z^2 \geq \rho_0^2 \end{cases} \tag{3}$$

where $\rho_0 = \sqrt{15 B_0/2c}$ and where B_0 is the uniform magnetic field at infinity. This solution will be obtained in the model problem when the plasma radius becomes very small so that the cylindrical wall is effectively very far away, i.e., when c is very large. This means there exists a two-dimensional family of solutions whose large c limit is given by Eq. (3) (B_0 is replaced by $2\psi_w/a^2$). As c is decreased, the equilibria should become larger and finally approach a final state at some critical value of c.

We tried to compute these larger equilibria by finite difference methods on a mesh, but ran into difficulties. We conjecture that the sharp-edged current distribution and the free-boundary non-linearity in the problem were what caused our iteration methods to converge to states that were not in fact equilibria. To overcome these difficulties we completely reformulated the problem. Using the Green's theorem for Δ^*, the Grad-Shafranov equation may be inverted to obtain the equation

$$\psi = -\int G(r,r';z,z') p'(\psi)\, r' dr' dz' + \frac{1}{2\pi} \oint \psi \frac{\partial G}{\partial r'} \frac{d\ell'}{r'^2} \tag{4}$$

where G satisfies $\Delta^* G = r\delta(r-r')\delta(z-z')$ and $G = 0$ if $r, r' = a$. The Green's function is given by the expression

$$G(r,r';z,z') = \frac{rr'}{\pi} \int_0^\infty \cos k(z-z') I_1(kr) I_1(kr') \frac{K_1(ka)}{I_1(ka)} dk - \frac{\sqrt{rr'}}{2\pi} Q_{1/2}\left(\frac{r^2+r'^2+(z-z')^2}{2rr'} \right) \tag{5}$$

Formulating the problem this way has the advantage that it is not necessary to compute finite differences across the separatrix where the current density may be discontinuous; the integration is taken over the region where there is current and the Green's function takes care of the vacuum field. Doing the integrations accurately requires a fine mesh; if a general pressure profile were used, iteration would be necessary to find ψ inside the separatrix, and this method might be very expensive. But for our model problem, there is no ψ dependence on the right-hand side of Eq. (4) except for the shape of the separatrix. Since the separatrix is given by $\psi = 0$, Eq. (4) can be used to obtain the following non-linear equation for the separatrix.

$$c \int_\Omega G\, r' dr' dz' + \psi_w \frac{r^2}{a^2} = 0 \tag{6}$$

where Ω is the region in the $r'-z'$ plane bounded by the separatrix. We do the integrations in spherical coordinates and represent the separatrix as an expansion in even-order Legendre polynomials as follows.

$$\rho(x) = \Sigma_{n=1}^{N} a_n P_{2n}(x) \tag{7}$$

where $\rho(x)$ is the spherical radius of the separatrix at the polar angle $\theta = \cos^{-1}(x)$. The problem is solved when the a_n's are determined.

When Eq. (6) is solved for very large c the small radius spherical solution is recovered. As c is decreased the solutions remain practically spherical until the radius of the solution at z=0 becomes greater than about .6ε; as c is decreased further, the solutions become prolate and racetrack-like in shape. As c approaches the one-dimensional critical value, the elongation evidently becomes infinite, and this two-dimensional solution branch connects with the 1-dimensional solution branches right at their bifurcation point. Figure 2 shows the separatrix shapes for a sequence of values of c approaching the critical value, Fig. 3 shows the elongation of the solutions as a function of c, and Fig. 4 shows the flux plot of a long equilibrium. For ratios of the separatrix half-length (z_{sep}) to midplane radius (r_{sep}) greater than about 4, a prohibitive number of a_n's are required. Hence, the exact behavior of r_{sep}/z_{sep} at the critical c is uncertain. Nonetheless, the elongation does appear to be sharply singular there; only in a relatively narrow region in c are very long equilibria obtained. Since these elongated equilibria lie near a bifurcation point, it should be a very delicate matter to compute them by standard numerical methods.

We have found up to four solutions of our model problem given a value of c. There may be other solution branches, but this two-dimensional branch contains elongated racetrack equilibria of the type observed in experiments.

Finally, we note that there are many pressure profiles whose one-dimensional solutions lie on two connected branches like that represented for the model problem in Fig. 1. In particular, if $p' = cf(\psi/\psi_0)$ where ψ_0 is the maximum value of ψ in the plasma, two one-dimensional solution branches parametrized by c are often obtained. We conjecture that for all such pressure profiles there exists at least one two-dimensional solution branch parametrized by c that becomes infinitely long as c approaches the one-dimensional critical value from above. If this conjecture is true, it is possible to obtain some information about the desired elongated equilibrium by solving the much simpler one-dimensional problem and examining its critical solution. The critical solution will have nearly the same average radius, trapped flux, radial magnetic field profile, and radial pressure profile as an elongated equilibrium with the same pressure profile.

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FIGURE CAPTIONS

Fig. 1 The $\psi=0$ radius is displayed as a function of c for the one-dimensional solution of the model problem.

Fig. 2 The separatrix shapes for a sequence of values of ca^4/ψ_w are shown: (a) 60, (b) 40, (c) 36, (d) 34.

Fig. 3 The elongation, z_{sep}/r_{sep}, is displayed as a function of c.

Fig. 4 A flux plot for the case $ca^4/\psi_w = 34$ is shown.

Figure 1

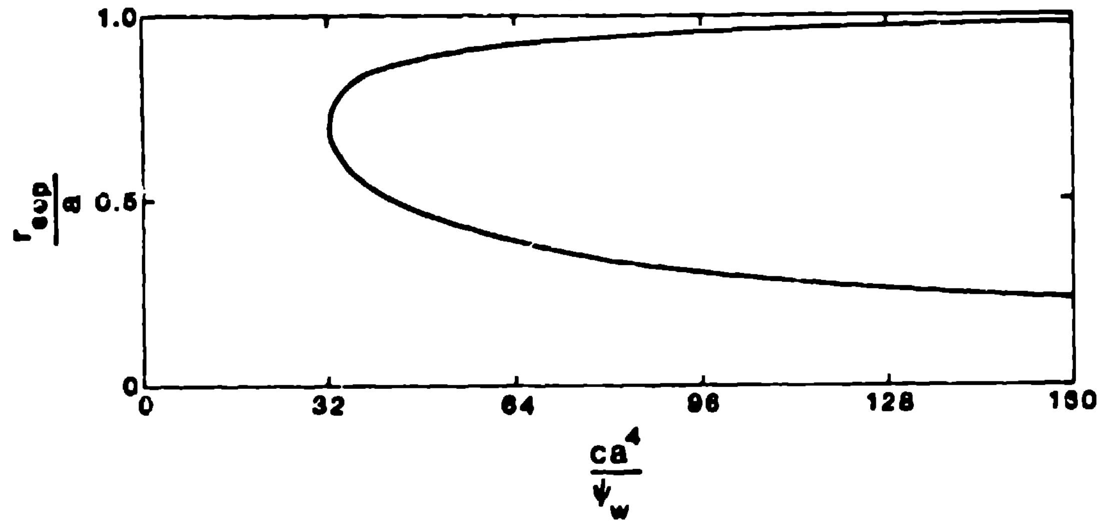

Figure 2

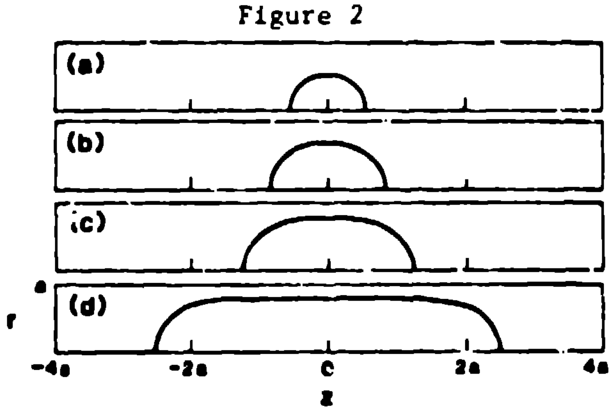

Figure 3

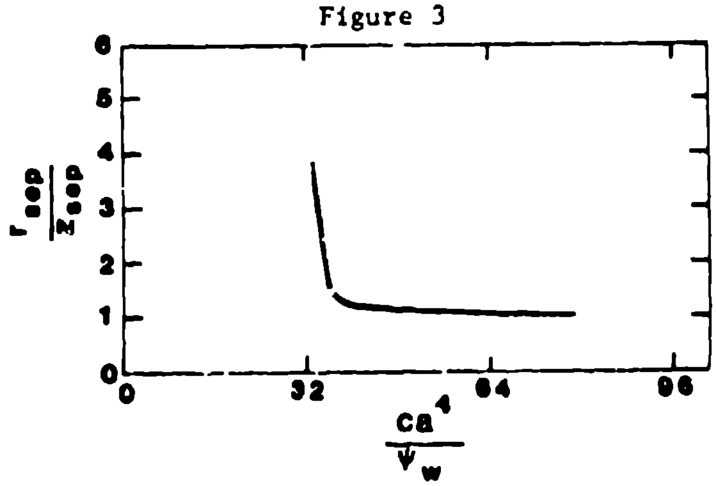

Figure 4